Math 8 Homework 3

1 Examples of Functions

(a) Each of the following is describing a function informally. Identify the domain and codomain of each function.

(i) For a natural number n, let $\sigma(n)$ be the sum of its positive divisors.

(ii) The map $x \mapsto x^2$ is a useful example in calculus.

(iii) Any set $S \subseteq \mathbb{R}$ has an outer measure $m^*(S)$, which is either a nonnegative real number or ∞.

(iv) The 2×3 matrix A can multiply any vector in $\mathbb{R}^3$. The result is a vector in the plane.

(v) The cross product on $\mathbb{R}^3$ gives a vector perpendicular to two given ones.

(vi) Given any continuous function $f \in C(\mathbb{R})$ we can compute its limit as $x \to 0$.

(vii) A sequence of real numbers $x_1, x_2, x_3, \ldots$

(b) Let S be a set. Suppose the function $\phi : \mathcal{P}(S) \to \mathbb{R}$ is *additive*; that is, for any $A, B \subseteq S$ that satisfy $A \cap B = \varnothing$, we have $\phi(A \cup B) = \phi(A) + \phi(B)$.

(i) Prove that $\phi(\varnothing) = 0$.

(ii) Prove that for any sets $A, B \subseteq S$ we have $\phi(A \cup B) = \phi(A) + \phi(B) - \phi(A \cap B)$.

2 Injectivity and Surjectivity

(a) Give an example of functions and sets $f : A \to B$ and $g : B \to C$ so that

(i) f is surjective, but $g \circ f$ is not.

(ii) g is surjective, but $g \circ f$ is not.

(iii) $g \circ f$ is surjective, but f is not.

(iv) g is injective, but $g \circ f$ is not.

(v) f is injective, but $g \circ f$ is not.

(vi) $g \circ f$ is injective, but g is not.

(b) Suppose that $f : A \to B$ and $g : B \to C$ are functions such that $g \circ f$ is surjective. Prove that g is also.

(c) Suppose that $f : A \to B$ and $g : B \to C$ are functions such that $g \circ f$ is injective. Prove that f is also.

For the following problems, recall the mean–value theorem.

Theorem (Mean–Value)**.** *Let $f : (a, b) \to \mathbb{R}$ be differentiable. Given points $x > y$ within (a, b) there is a c between x and y so that*

$$f(x) - f(y) = f'(c)(x - y).$$

(d) A function $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere. Prove that if f' is never zero, then f is injective.

(e) A differentiable function $f : (0, 1) \to \mathbb{R}$ satisfies $|f'(x)| \leq 1$ for all x. Prove that f is not surjective.

3 Images of Sets

(a) Let $f : A \to B$ be an arbitrary function and $C, D \subseteq A$.

(i) Prove that $f(C \cup D) = f(C) \cup f(D)$.

(ii) Prove that $f(C \cap D) \subseteq f(C) \cap f(D)$.

(iii) Give an example wherein $f(C \cap D) \neq f(C) \cap f(D)$.

(b) Let $f : A \to B$ be an arbitrary function and $E, F \subseteq B$.

(i) Prove that $f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F)$.

(ii) Prove that $f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$.